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Prolongation structures and nonlinear evolution equations in two spatial dimensions: A general class of equations

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Abstract. A new method of generating classes of equations which possess an inverse scattering formulation in two spatial dimensions is proposed and developed in detail.

1. Introduction

The success of the inverse scattering method (Gardener *et al* 1967) for finding exact solutions to specific equations in one time and one spatial dimension has led to a search for similar equations in two spatial dimensions which have such an inverse scattering formulation (Ablowitz and Haberman 1975a, b, Morris 1976, 1977, 1978). For the two-dimensional case it is well known that most of the equations soluble by the inverse scattering method can be derived from the equations (Ablowitz *et al* 1974; AKNS)

$$A_x = qC - rB \tag{1.1}$$

$$q_t = B_x + 2(Aq + \lambda iB) \tag{1.2}$$

$$r_t = C_x + 2(Ar + \lambda iC) \tag{1.3}$$

where λ is a constant. We will refer to these as the AKNS equations. If we are seeking a generalisation of these equations to two spatial dimensions we clearly require that these AKNS equations will result when we restrict ourselves to x and t dependence alone. This may not appear in the most general case, but must be so in the special cases when generalisation is possible. Consequently we will choose a set of forms initially which have that potentiality within them and hope that when we apply the methods developed in previous papers (Morris 1976, 1977) everything will go through smoothly. In § 2 we introduce a basic ideal of forms which we then generalise by using a new technique to a much larger matrix system. The equations equivalent to this ideal are the time-independent form of the generalised equations we are trying to determine. In the following section time-derivative terms are added to obtain the required three-dimensional equations and in § 4 we establish an inverse scattering problem for the

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resultant equations. The scope of the methods introduced in this paper is large; several directions in which the examples of this paper may be generalised are indicated and will be developed in future work.

2. The basic ideal

The set of differential two-forms

$$\alpha_1 = \mathrm{d}S \wedge \mathrm{d}x + \mathrm{d}A \wedge \mathrm{d}y + (rB - qC)\mathrm{d}x \wedge \mathrm{d}y \tag{2.1}$$

$$\alpha_2 = \mathrm{d}A \wedge \mathrm{d}x + \mathrm{d}S \wedge \mathrm{d}y \tag{2.2}$$

$$\alpha_3 = \mathrm{d}B \wedge \mathrm{d}y + \mathrm{d}q \wedge \mathrm{d}x + 2(Aq - SB)\, \mathrm{d}x \wedge \mathrm{d}y \tag{2.3}$$

$$\alpha_4 = \mathrm{d}C \wedge \mathrm{d}y + \mathrm{d}r \wedge \mathrm{d}x - 2(Ar - SC)\, \mathrm{d}x \wedge \mathrm{d}y \tag{2.4}$$

span a closed ideal of forms equivalent to the partial differential equations

$$S_y = A_x + rB - qC \tag{2.5}$$

$$A_y = S_x \tag{2.6}$$

$$q_y = B_x + 2(Aq - SB) \tag{2.7}$$

$$r_y = C_x - 2(Ar - SC). \tag{2.8}$$

When $S = -\mathrm{i}\lambda$ these equations reduce to the time-independent form of the AKNS equations.

A prolongation structure may be easily established for the ideal (2.1)–(2.4) using standard methods (Wahlquist and Estabrook 1975, 1976, Dodd and Gibbon 1977, 1978) and takes the form

$$\Omega = \mathrm{d}\zeta + (Sx_1 + Ax_2 + qx_3 + rx_4)\, \mathrm{d}x + (Ax_1 + Sx_2 + Bx_3 + Cx_4)\, \mathrm{d}y \tag{2.9}$$

where the x_i satisfy the Lie bracket relations

$$[x_1, x_3] = 2x_3 \qquad [x_1, x_4] = -2x_4 \qquad [x_3, x_4] = x_1 \qquad [x_2, x_i] = 0 \qquad \forall i. \tag{2.10}$$

A two-dimensional representation of this algebra is given by

$$x_1 = \zeta^2 b_2 - \zeta^1 b_1 \qquad x_2 = -(\zeta^1 b_1 + \zeta^2 b_2) \tag{2.11}$$

$$x_3 = -\zeta^2 b_1 \qquad x_4 = -\zeta^1 b_2 \tag{2.12}$$

where $b_i = \partial/\partial\zeta^i$ and yields the linear prologation structure

$$\Omega = \mathrm{d}\zeta + F(S, A, q, r)\zeta\, \mathrm{d}x + G(S, A, B, C)\zeta\, \mathrm{d}y \tag{2.13}$$

with

$$F = \bar{x}_1 S + \bar{x}_2 A + \bar{x}_3 q + \bar{x}_4 r \tag{2.14}$$

$$G = \bar{x}_1 A + \bar{x}_2 S + \bar{x}_3 B + \bar{x}_4 C \tag{2.15}$$

where $\zeta = (\zeta^1, \zeta^2)^t$ and $\bar{x}_i$ are the matrices

$$\bar{x}_1 = -\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad \bar{x}_2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \bar{x}_3 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad \bar{x}_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{2.16}$$

We generalise the system by allowing the variables S, A, q and r to become matrix-valued and by replacing the prolongation forms Ω by $\hat{\Omega}$ defined by

$$\hat{\Omega} = \mathrm{d}\zeta + (\bar{x}_1 \otimes S + \bar{x}_2 \otimes S + \bar{x}_3 \otimes q + \bar{x}_4 \otimes r)\zeta \, \mathrm{d}x + (\bar{x}_1 \otimes A + \bar{x}_2 \otimes S + \bar{x}_3 \otimes B + \bar{x}_4 \otimes C)\zeta \, \mathrm{d}y \tag{2.17}$$

where $\zeta = (\zeta^1, \zeta^2, \zeta^3, \zeta^4)^t$.

This set of one-forms is easily shown to provide a prolongation structure for the equations

$$S_y = A_x + \tfrac{1}{2}(\{B, r\} - \{q, C\}) \tag{2.18}$$

$$A_y = S_x + \tfrac{1}{2}([B, r] - [q, C]) \tag{2.19}$$

$$q_y = B_x + \{A, q\} - \{S, B\} + [S, q] - [A, B] \tag{2.20}$$

$$r_y = C_x - \{A, r\} + \{S, C\} + [S, r] - [A, C] \tag{2.21}$$

where for any pair of square matrices U and V we have defined $\{U, V\} = UV + VU$ and $[U, V] = UV - VU$.

These equations have a far richer structure than the original system and form a suitably general base for extension to higher dimensions. An example of some importance is given by the particular choice of parametrisation

$$q = \begin{bmatrix} -A^* & -R^* \\ 0 & 0 \end{bmatrix} \qquad r = \begin{bmatrix} -A & -L \\ 0 & 0 \end{bmatrix} \tag{2.22}$$

$$B = \begin{bmatrix} 0 & -L^* \\ 0 & -A^* \end{bmatrix} \qquad c = \begin{bmatrix} 0 & R \\ 0 & A \end{bmatrix} \tag{2.23}$$

$$S = \begin{bmatrix} 0 & -\tfrac{1}{2}(\Phi - \Psi) \\ -1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -\tfrac{1}{2}(\Phi + \Psi) \\ 0 & 0 \end{bmatrix}. \tag{2.24}$$

With this form of the matrices q, r, B, C, S and A equations (2.18) and (2.19) become

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\Phi = -(RA^* + R^*A) \tag{2.25}$$

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\Psi = (LA^* + L^*A). \tag{2.26}$$

Equation (2.20) gives rise to the identities

$$A_y^* = (L + R)^* \tag{2.27}$$

$$A_x^* = (R - L)^* \tag{2.28}$$

$$R_y^* = L_x^* + A^*(\Phi - \Psi) \tag{2.29}$$

and finally equation (2.21) yields the relationships

$$A_y = (R + L) \tag{2.30}$$

$$A_x = (R - L) \tag{2.31}$$

$$L_y = -R_x + A(\Phi - \Psi). \tag{2.32}$$

Combining these equations together yields the underlying equations

$$\left(\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y^2}\right)A=2A(\Phi-\Psi) \tag{2.33}$$

$$\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right)\Phi=-\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(AA^*) \tag{2.34}$$

$$\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\right)\Psi=\frac{1}{2}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right)(AA^*) \tag{2.35}$$

which were used by Morris (1977) to construct a prolongation structure for the generalised nonlinear Schrödinger equation considered in that paper.

3. Generalising the equations to three dimensions

The method introduced by Morris (1976) can be easily extended to matrix-valued variables. The closed ideal of matrix-valued two-forms defined by

$$\alpha_1=\mathrm{d}S\wedge\mathrm{d}x+\mathrm{d}A\wedge\mathrm{d}y+\tfrac{1}{2}(\{B,r\}-\{q,C\})\,\mathrm{d}x\wedge\mathrm{d}y \tag{3.1}$$

$$\alpha_2=\mathrm{d}A\wedge\mathrm{d}x+\mathrm{d}S\wedge\mathrm{d}y+\tfrac{1}{2}([B,r]-[q,C])\,\mathrm{d}x\wedge\mathrm{d}y \tag{3.2}$$

$$\alpha_3=\mathrm{d}B\wedge\mathrm{d}y+\mathrm{d}q\wedge\mathrm{d}x+(\{A,q\}-\{S,B\}+[S,q]-[A,B])\,\mathrm{d}x\wedge\mathrm{d}y \tag{3.3}$$

$$\alpha_4=\mathrm{d}C\wedge\mathrm{d}y+\mathrm{d}r\wedge\mathrm{d}x+(\{S,C\}-\{A,r\}+[S,r]-[A,C])\,\mathrm{d}x\wedge\mathrm{d}y \tag{3.4}$$

can be generalised to the closed ideal of matrix-valued three-forms

$$\bar{\alpha}_i=\alpha_i\wedge\mathrm{d}t+\beta_i\wedge\mathrm{d}x\wedge\mathrm{d}y \qquad i=1,\ldots,q \tag{3.5}$$

provided that we can find two constant (4×4) matrices M and N having the properties

$$(dGM-dFN)=\sum_{i=1}^{4}\bar{x}_i\otimes\beta_i \tag{3.6}$$

$$[M,N]=0 \tag{3.7}$$

$$[G,M]+[N,F]=0.$$

If we seek a solution of these equations in the form

$$M=a\otimes b \qquad N=c\otimes d \tag{3.9}$$

with $[a,c]=0=[b,d]$ then (3.7) is automatically satisfied. Also, as the $\bar{x}_i$ are a basis for (2×2) matrices, $(dGM-dFN)$ will be expressible in the form required in (3.6).

If we choose

$$a=I \qquad \text{and} \qquad c=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

we then find that equations (3.8) reduce to

$$[S,d]=[S,b] \tag{3.10}$$

$$[A,d]=[A,b] \tag{3.11}$$

$$[B,b]=-\{q,d\} \tag{3.12}$$

$$[C, b] = \{r, d\}. \tag{3.13}$$

The choice $b = d$ automatically satisfies (3.10) and (3.11) and this leaves the subsidiary conditions

$$[B, b] = -\{q, b\} \tag{3.14}$$

Equation (3.6) becomes

$$(dGM - dFN) = \bar{x}_3 \otimes d[(B+q)b] + \bar{x}_4 \otimes d[(C-r)b]$$

and we can see that for this choice of M and N

$$\beta_1 = \beta_2 = 0 \tag{3.16}$$

and

$$\beta_3 = d(B+q)b \qquad \beta_4 = d(C-r)b. \tag{3.17}$$

Thus the generalised equations are given by

$$S_y = A_x + \tfrac{1}{2}(\{B, r\} - \{q, C\}) \tag{3.18}$$

$$A_y = S_x + \tfrac{1}{2}([B, r] - [q, C]) \tag{3.19}$$

$$q_y = B_x + \{A, q\} - \{S, B\} + [S, q] - [A, B] + (B_t + q_t)b \tag{3.20}$$

$$r_y = C_x - \{A, r\} + \{S, C\} + [S, r] - [A, C] + (C_t - r_t)b \tag{3.21}$$

together with the constraints of equations (3.14) and (3.15) that

$$[B, b] = -\{q, b\} \qquad \text{and} \qquad [C, b] = \{r, b\}. \tag{3.22}$$

For the parametrisation (2.22)–(2.24) of the two-dimensional case we can choose

$$b = \tfrac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix} \tag{3.23}$$

and we obtain the nonlinear equations

$$-\mathrm{i}\frac{\partial A}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)A = 2A(\Phi - \Psi) \tag{3.24}$$

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\Phi = -\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(AA^*) \tag{3.25}$$

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\Psi = \frac{1}{2}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)(AA^*) \tag{3.26}$$

which are a form of generalised Schrödinger equation and which have been analysed elsewhere (Morris 1977).

4. A prolongation structure for the generalised equations

We now have sufficient information in the matrices M and N to construct a prolongation structure for the equations (3.18)–(3.21) which generalises that given in equation (2.17) for equations (2.5)–(2.8). We can include a parameter to serve as a scattering eigenvalue by simply replacing F by $F - \lambda I$. More general methods have been discussed elsewhere (Morris 1977), and in general depend upon an internal symmetry of a particular equation rather than existing for a general class. Further details concerning this particular nonlinear Schrödinger equation can be found in recent work by the author (Morris 1978). We will first examine the above substitution and then show how to include a more useful scattering parameter by utilising a symmetry of our general system. Our prolongation forms become

$$\Omega^1 = d\zeta^1 \wedge dt - [(S+A+\lambda)\zeta^1 + q\zeta^2]\, dx \wedge dt - [(A+S)\zeta^1 + B\zeta^2]\, dy \wedge dt + b(dx+dy) \wedge d\zeta^1 - (B+q)b\zeta^2\, dx \wedge dy \tag{4.1}$$

and

$$\Omega^2 = d\zeta^2 \wedge dt - \{r\zeta^1 + [(A-S)+\lambda I]\zeta^2\}\, dx \wedge dt - [c\zeta^1 + (S-A)\zeta^2]\, dy \wedge dt + b(dx-dy) \wedge d\zeta^2 - (C-r)b\zeta^1\, dx \wedge dy \tag{4.2}$$

where

$$\zeta^1 = (\xi^1, \xi^2)^t \qquad \zeta^2 = (\xi^3, \xi^4)^t.$$

The inverse scattering problem which is obtained from the equations

$$\Omega^1 = \Omega^2 = 0 \tag{4.3}$$

is given by

$$\zeta^1_x = [(S+A)+\lambda I]\zeta^1 + q\zeta^2 - b\zeta^1_t \tag{4.4}$$

$$\zeta^1_y = (A+S)\zeta^1 + B\zeta^2 - b\zeta^1_t \tag{4.5}$$

$$\zeta^2_x = r\zeta^1 + [(A-S)+\lambda I]\zeta^2 - b\zeta^2_t \tag{4.6}$$

$$\zeta^2_y = C\zeta^1 + (S-A)\zeta^2 + b\zeta^2_t. \tag{4.7}$$

Even though the scattering parameter is included in a trivial way, one obtains the interesting result that if one restricts to the $y-t$ plane one obtains an inverse scattering problem equivalent to the normal nontrivial AKNS scattering problem. Details of this reduction in the case of the nonlinear Schrödinger equation can be found in earlier work (Morris 1977). We can obtain a generally more useful inverse scattering by utilising a symmetry of our equation system (3.15)–(3.22). One easily sees that replacing A by $A+\lambda b$ leaves the system unchanged and so by making that replacement in the prolongation forms we can obtain the inverse scattering problem

$$\zeta^1_x = (S+A)\zeta^1 + \zeta^2 + b(\lambda\zeta^1 - \zeta^1_t) \tag{4.8}$$

$$\zeta^2_x = (A-S)\zeta^2 + r\zeta^1 + b(\lambda\zeta^2 - \zeta^2_t) \tag{4.9}$$

$$\zeta^1_y = (A+S)\zeta^1 + B\zeta^2 + b(\lambda\zeta^1 - \zeta^1_t) \tag{4.10}$$

$$\zeta^2_y = (S-A)\zeta^2 + C\zeta^1 - b(\lambda\zeta^2 - \zeta^2_t). \tag{4.11}$$

A consideration of this system in the case of the generalised nonlinear Schrödinger equation can be found in earlier work (Morris 1977). Clearly many more sets of equations which have the special property of possessing an inverse scattering formulation may be obtained by our methods. Different matrix representations and different dimensions for the new matrix-valued variables may be used. We have presented a practical method which we hope to place in a more general algebraic setting in future work.

References

Ablowitz M J and Haberman R 1975a *J. Math. Phys.* **16** 2301
—— 1975b *Phys. Rev. Lett.* **35**
Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Studies Appl. Math.* **53** 249
Dodd R and Gibbon J 1977 *Proc. R. Soc.* A **358** 287
—— 1978 *Proc. R. Soc.* A **359** 411
Gardener C S, Green J M, Kruskall M and Miura R 1967 *Phys. Rev. Lett.* **39** 1095
Morris H C 1976 *J. Math. Phys.* **17** 1870
——1977 *J. Math. Phys.* **18** 285
—— 1978 *Proc. Ames Sem. on Differential and Algebraic Geometry for Control Engineers* (Brookline: Math. Sci. Press)
Wahlquist H and Estabrook F 1975 *J. Math. Phys.* **16** 1
——1976 *J. Math. Phys.* **17** 1993